X-wave-mediated instability of plane waves in Kerr media

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Plane waves in Kerr media spontaneously generate paraxial X-waves (i.e., nondispersive and nondiffractive pulsed beams) that get amplified along propagation. This effect can be considered to be a form of conical emission (i.e., spatiotemporal modulational instability), and can be used as a key for the interpretation of the out-of-axis energy emission in the splitting process of focused pulses in normally dispersive materials. A new class of spatiotemporal localized wave patterns is identified. X-wave instability and nonlinear X waves are predicted for both focusing and defocusing media and are expected in periodical Bose condensed gases.

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The dynamics of focused femtoseconds pulses (FFP) in optically nonlinear media has a fundamental importance and it is relevant in all the applications of ultrafast optics. The basic mechanism, when dealing with the propagation in normally dispersive materials, is the splitting of the pulse, which has been originally predicted more than ten years ago [1,2] and recently reconsidered, due to the development of the physics of FFP [3–13].

If z is the direction of propagation, t is the time in the reference frame where the pulse is still and is $r=\sqrt{x^2+y^2}$ the radial cylindrical coordinate, this process can be roughly divided into a series of steps. They describe the propagation of a Gaussian (in space and time) pulse, which travels in a focusing medium, and undergoes relevant reshaping, due to the interplay of diffraction, dispersion, and the Kerr effect (when the peak power is sufficiently greater than the critical power P_c for self-focusing [1,2]): (1) the out-of-axis energy is focused towards $r=0$ around $t=0$; (2) the pulse at $r=0$ is compressed; (3) lobes appear in the on-axis temporal spectrum; and (4) the pulse splits in the time domain.

Before the breakup, relevant out-of-axis energy emission and redistribution occurs, as originally described by Rothenberg in Ref. [2]. Looking at the spatiotemporal profile an X shape (or hyperbola) is observed before the splitting (see, e.g., figures in Refs. [10,12]). This process has been theoretically described by different approaches [10,12,14,15], and the onset of an X shape can be ultimately related to the hyperbolic characteristics over which small perturbations evolve [16]. This can be checked, for example, by the hydrodynamical approach to the nonlinear Schrödinger equation [17,18].

Recently, attention has been devoted to the existence of X waves in optically nonlinear media. The latter are self-trapped (i.e., nondiffracting and nondispersive) waves that are well known in the field of linear propagation in acoustics [19–22] and in electromagnetism [23–30]. The simplest X wave has the shape of a double cone, or clepsydra, that appears as an X when, e.g., a section is plotted in the plane (x,t), and as V in the plane (r,t), as shown in Fig. 1. Optical X waves in nonlinear media have been theoretically predicted in Ref. [31] and experimental results, with a direct observation of the conical spatiotemporal shape, have confirmed their existence and generation in crystals for second-harmonic generation (SHG) [32,33]. In Ref. [34], it has been theoretically shown how, during nondepleted-pump SHG, the phase matched spatiotemporal harmonics let the SH beam become an X wave [47].

Conical emission (CE), or spatiotemporal modulational instability (MI) ([14,35–37]), has been addressed in Ref. [38] as a basic mechanism underlying the spontaneous formation of an X wave, and as a foundation for understanding the splitting, in Ref. [10]. In this paper, the formation of X waves in Kerr media (or in quadratic media, in the regime where they mimic cubic nonlinearity, see, e.g., the chapter after Torruellas, Kivhsar, and Stegeman in Ref. [39] and Ref. [40]) is considered. A new form of instability of plane waves can be introduced by directly involving self-localized spatiotemporal wave patterns. The process strongly resembles CE, i.e., the amplification of plane waves from noise, but in this case X waves, instead of periodical patterns, emerge from the breakup of an unstable pump beam. This mechanism is responsible for the first stage of pulse splitting, i.e., the out-of-axis energy redistribution, such as MI breaks a continuous wave signal into a periodical pattern of solitons [41].

The wave equation describing the propagation in nonlin-

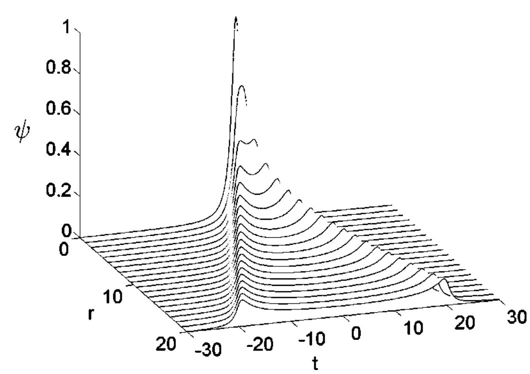

FIG. 1. Three-dimensional (3D) plot of ψ_X, the simplest radial-symmetry X wave $(\Delta=1)$.

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ear Kerr media, whose refractive index n obeys the law $n=n_0+n_2I$, with I as the optical intensity, in the framework of the paraxial and the slowly varying envelope approximation, is written as

$$i\partial_Z A+ik'\partial_T A+\frac{1}{2k}\nabla^2_{XY}A-\frac{k''}{2}\partial_{TT}A+\frac{kn_2}{n_0}|A|^2A=0, \quad (1)$$

where A is normalized such that $|A|^2=I$ and $k''>0$ (i.e., the medium is normally dispersive). X,Y,Z,T are the real world variables, λ the wavelength, $n(\lambda)$ the refractive index, and $k(\lambda)=2\pi n(\lambda)/\lambda$. Equation (1) can be cast in the nondimensional form as

$$i\partial_z u+\Delta_\perp u-\partial_{tt}u+\chi|u|^2u=0 \quad (2)$$

by defining $z=Z/Z_{df}$ with $Z_{df}=2kW_0^2$ and W_0^2 a reference beam waist; $\Delta_\perp\equiv\partial_{xx}+\partial_{yy}$ the tranverse Laplacian with $(X,Y)=W_0(x,y)$; $t=(T-Z/V_g)/T_0$ the retarded time in the frame traveling at group velocity $V_g=1/k'$ in units of $T_0=(k''Z_{df}/2)^{1/2}$. The optical field envelope A is given by $A=A_0u$, with $A_0=(n_0/k|n_2|Z_{df})^{1/2}$. $\chi>0$ $(\chi<0)$ identifies a focusing (defocusing) medium to be $n_2>0$ $(n_2<0)$.

Let us start by considering paraxial linear (i.e., $\chi=0$) X-wave solutions of Eq. (2) (paraxial X waves have been considered in detail in Ref. [42]), defined by

$$(\Delta_\perp-\partial_{tt})\psi=0. \quad (3)$$

Introducing the complex variable $v=(\Delta-it)^2+r^2$, with Δ a real-valued arbitrary coefficient, we have from Eq. (3)

$$6\partial_v\psi+4v\partial_{vv}\psi=0, \quad (4)$$

from which $\psi=C_1/\sqrt{v}+C_2$. C_1 and C_2 are arbitrary complex coefficients. Note that both the real and the imaginary parts of this solution are real-valued X-wave profiles. The former being the simplest X wave, given by (the branch cut for the square root is along the negative real axis)

$$\psi_X\equiv\mathrm{Re}\left(\frac{1}{\sqrt{v}}\right)=\mathrm{Re}\left(\frac{1}{\sqrt{(\Delta-it)^2+r^2}}\right). \quad (5)$$

Remarkably, ψ_X still holds when referring to the Helmoltz equation, instead of the paraxial wave equation (see, e.g., Ref. [43]). Its plot is given in Fig. 1.

X-wave instability can be introduced in the same way as MI, i.e., by perturbing the plane-wave solution of Eq. (2): $a=a_P\equiv a_0\exp(ia_0^2z)$, with a_0 a real-valued constant. Assuming $a=[a_0+\epsilon(x,y,t,z)]\exp(ia_0^2z)$ we have, at first order in ϵ,

$$i\partial_z\epsilon+(\Delta_\perp-\partial_{tt})\epsilon+\chi a_0^2(\epsilon^*+\epsilon)=0. \quad (6)$$

Writing $\epsilon=\epsilon(r,t,z)=\psi_X(r,t)\mu(z)$ gives

$$i\partial_z\mu+\chi a_0^2(\mu+\mu^*)=0. \quad (7)$$

The perturbation is thus

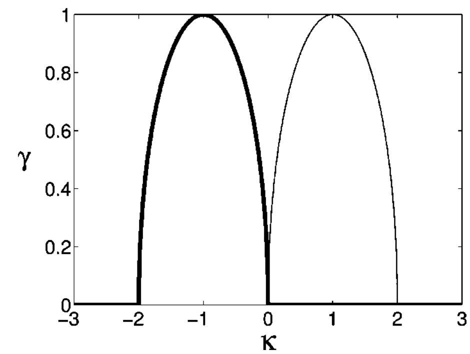

FIG. 2. Gain γ vs κ, the parameter identifying the generalized X wave, for focusing (thick line) and defocusing (thin line) media $(a_0=1)$.

$$\epsilon=[\alpha+i(\beta+2\chi a_0^2\alpha z)]\psi_X(r,t), \quad (8)$$

with α and β arbitrary real-valued constants. Equation (8) represents an X wave which grows linearly along propagation (independent of the sign of χ), with amplification given by the intensity of the plane wave a_P. This situation strongly resembles MI, where plane waves are exponentially amplified at the expense of the pump beam, with a gain determined by the pump intensity. For this reason, it is natural to refer to this process as X-wave instability. As for MI, the amplified wave can be artificially externally feeded, or it can be generated by noise [48]. Note also that X-wave instability can be triggered by conical emission. Indeed, as shown in Ref. [38], the latter generates the required spectrum to form an X wave, which eventually gets amplified, as discussed above.

The previous treatment can be generalized in several ways. In particular, it is possible to show that exponential amplification of X-wave-like beams can be attained. It is necessary to broaden the definition of X waves, i.e., Eq. (3), by introducing the equation

$$(\Delta_\perp-\partial_{tt})\psi=\kappa\psi, \quad (9)$$

with κ a real constant ($\kappa\neq0$ in the following). Assuming $\epsilon=\mu(z)\psi(r,t)+\nu(z)^*\psi(r,t)^*$ in Eq. (6) and setting to zero the coefficients of ψ and ψ^*, the following linear system is obtained:

$$\begin{aligned} &i\partial_z\mu+\kappa\mu+a_0^2\chi(\mu+\nu)=0,\\ &-i\partial_z\nu+\kappa\mu+a_0^2\chi(\mu+\nu)=0. \end{aligned} \quad (10)$$

If $(\mu,\nu)=(\hat\mu,\hat\mu)\exp(\gamma z)$, we have

$$\begin{bmatrix} i\gamma+\kappa+\chi a_0^2 & \chi a_0^2 \\ \chi a_0^2 & -i\gamma+\kappa+\chi a_0^2\end{bmatrix}\begin{bmatrix}\hat\mu\\ \hat\nu\end{bmatrix}=0. \quad (11)$$

The solvability condition yields the allowed values for the gain $\gamma=\pm\sqrt{-\kappa(\kappa+2\chi a_0^2)}$. The perturbation grows along z if the following inequality is satisfied: $-\kappa(\kappa+2\chi a_0^2)>0$. Thus, for a focusing (defocusing) medium, generalized X waves with $-2a_0^2<\kappa<0$ $(0<\kappa<2a_0^2)$ are exponentially

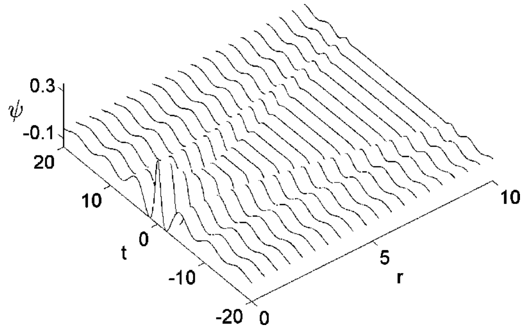

FIG. 3. 3D plot of the generalized X wave ψ_κ when $\kappa=\Delta=1$.

amplified. The gain vs κ, $\gamma=\sqrt{-\kappa(\kappa+2\chi a_0^2)}$, plotted in Fig. 2, has a maximum value which corresponds to the most exploding self-localized packet.

Now the question arises as to whether or not Eq. (9) admits solutions resembling X waves. Closed forms can be found by proceeding as before: In terms of v, Eq. (9) becomes

$$6\partial_v\psi+4v\,\partial_{vv}\psi=\kappa\psi, \tag{12}$$

whose general solution is

$$\psi=C_1\frac{\exp(-\sqrt{\kappa v})}{\sqrt{v}}+C_2\frac{\exp(\sqrt{\kappa v})}{\sqrt{v}}. \tag{13}$$

A real-valued, localized, generalized X wave is given by

$$\psi_\kappa=\mathrm{Re}\left(\frac{\exp(-\sqrt{\kappa v})}{\sqrt{v}}\right). \tag{14}$$

Equation (14) depends on two parameters Δ and κ; while the first determines the decay constant as going far from the origin in the (r,t) plane, the latter completely changes the shape of the wave. Examples for $\Delta=1$ and $\kappa=1$ are shown in Figs. 3 and 4 for $\Delta=1$ and $\kappa=-1$, respectively. Note that they are similar to the Bessel pulse beams described in Ref. [29]. To clarify the differences, we observe that the spatiotemporal spectrum develops around the curve $\omega^2=k_\perp^2+\kappa$, with ω the angular frequency corresponding to t and $k_\perp$ the transversal wave number. In Fig. 5, the spectrum for different κ, with $\kappa=0$ corresponding to the simplest X wave ψ_X, is shown. The appearance of this lines in the spatiotemporal far field is a clear signature of the X-wave instability, and can be directly observed in experiments. Note that the spectrum resemble the shape of the wave in the physical space as a consequence of their propagation invariance.

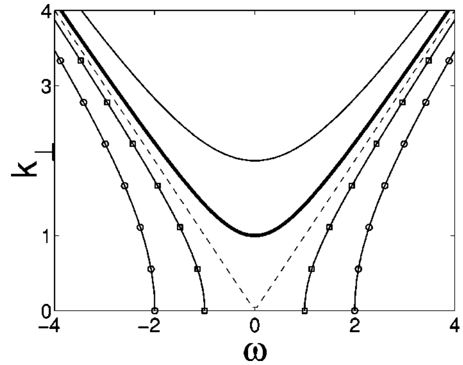

FIG. 5. Spatiotemporal spectrum (transversal wave vector vs angular spectrum) for different generalized X waves (thin line, $\kappa=-2$; thick line, $\kappa=-1$; squares, $\kappa=1$; circles $\kappa=4$). The dashed line is the spectrum of the simplest X wave ($\kappa=0$).

To show that X-wave instability can actually be observed in the experiments Eq. (2) is numerically solved. A Gaussian input beam is considered, $A=A_0\exp[-R^2/(2W_0^2)-T^2/(2T_p^2)]$, whose intensity profile full width at half maximum spot and duration are 70 μm and 100 fs. Equation (2) is integrated with reference to fused silica, with $\lambda=800$ nm ($n_0=1.5$, $n_2=2.5\times10^{-20}$ $k''=360\times10^{-28}$, SI units), peak power $P=1.5P_c$, being $P_c=(0.61\lambda)^2\pi/(8n_0n_2)\cong2.6$ MW the critical power for self-focusing. In Fig. 6, the spatiotemporal profile and the spectrum (in log scale) are shown after three diffraction lengths $L_{df}=\pi n_0W_0^2/\lambda$. Clearly, an X-like profile is formed and the spectrum shows the features in Fig. 5.

In conclusion, it is observed that a plane wave in Kerr media gives rise to linear and exponential amplification of X waves, thus leading to a significant beam reshaping. A new class of X waves is involved in this nonlinear process. The reported analysis provides insights for the interpretation of pulse splitting of focused femtosecond beams, and related phenomena, in the same way as MI is relevant for soliton generation. Indeed, the spontaneous formation of an X wave can be another explanation for the out-of-axis energy redistribution typically observed. Notably, X-wave instability is

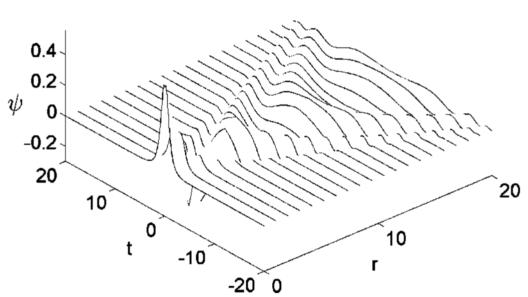

FIG. 4. 3D plot of the generalized X wave ψ_κ when $\kappa=-\Delta=1$.

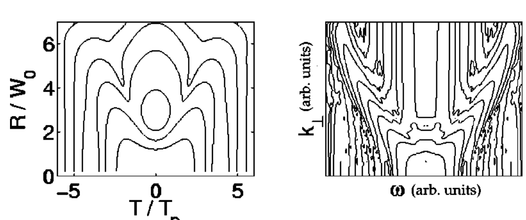

FIG. 6. Results of numerical integration of Eq. (2) after three diffraction lengths. (Left) Level plots of $10\log_{10}(|u|^2)$ and (right) level plots of the square modulus of the spectrum in log scale of u.

expected for both focusing and defocusing media, and can be an effective approach for the controlled generation of nondispersive and nondiffractive pulses.

These results appear to be susceptible to several generalizations, such as considering quadratric nonlinearity or vectorial effects, and have implications in all the fields encompassing nonlinear wave propagation, such as acoustics, hydrodynamics, and plasma physics. For example, X-wave instability (as well as nonlinear X waves) is also expected in periodical Bose-Einstein condensates where Eq. (2) holds, t being the direction of periodicity, in the presence of negative effective mass [44,45].

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[1] P. Chernev and V. Petrov, Opt. Lett. **17**, 172 (1992).
[2] J. Rothenberg, Opt. Lett. **17**, 583 (1992).
[3] J.K. Ranka, R.W. Schirmer, and A.L. Gaeta, Phys. Rev. Lett. **77**, 3783 (1996).
[4] L. Berge, J.J. Rasmussen, E.G. Shapiro, and S.K. Turitsyn, J. Opt. Soc. Am. B **13**, 1879 (1996).
[5] T. Brabec and F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).
[6] M. Trippenbach and Y.B. Band, Phys. Rev. A **56**, 4242 (1997).
[7] J.K. Ranka and A.L. Gaeta, Opt. Lett. **23**, 534 (1998).
[8] S.A. Diddams, H.K. Eaton, A.A. Zozulya, and T.S. Clement, Opt. Lett. **23**, 379 (1998).
[9] A.A. Zozulya and S.A. Diddams, Opt. Express **4**, 336 (1999).
[10] A.G. Litvak, V.A. Mironov, and E.M. Sher, Phys. Rev. E **61**, 891 (2000).
[11] H.S. Eisenberg, R. Morandotti, Y. Silberberg, S. Bar-Ad, D. Ross, and J.S. Aitchison, Phys. Rev. Lett. **87**, 043902 (2001).
[12] L. Berge, K. Germaschewski, R. Grauer, and J.J. Rasmussen, Phys. Rev. Lett. **89**, 153902 (2002).
[13] H. Ward and L. Berge, Phys. Rev. Lett. **90**, 053901 (2003).
[14] G.G. Luther, A.C. Newell, and J.V. Moloney, Physica D **74**, 59 (1994).
[15] G. Fibich, V.M. Malkin, and G.C. Papanicolaou, Phys. Rev. A **52**, 4218 (1995).
[16] T. Levi-Civita, *Caratteristiche dei Sistemi Differenziali e Propagazione Ondosa* (Zanichelli, Bologna, Italy, 1931).
[17] K. Staliunas, Phys. Rev. A **48**, 1573 (1993).
[18] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[19] R.W. Ziolkowski, D.K. Lewis, and B.D. Cook, Phys. Rev. Lett. **62**, 147 (1989).
[20] J. Lu and J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 441 (1992).
[21] P.R. Stepanishen and J. Sun, J. Acoust. Soc. Am. **102**, 3308 (1997).
[22] J. Salo, J. Fagerholm, A.T. Friberg, and M.M. Salomaa, Phys. Rev. Lett. **83**, 1171 (1999).
[23] J. Durnin, J.J. Miceli, and J.H. Eberly, Phys. Rev. Lett. **58**, 1499 (1987).
[24] E. Recami, Physica A **252**, 586 (1998).
[25] P. Saari and K. Reivelt, Phys. Rev. Lett. **79**, 4135 (1997).
[26] H. Sonajalg, M. Ratsep, and P. Saari, Opt. Lett. **22**, 310 (1997).
[27] K. Reivelt and P. Saari, J. Opt. Soc. Am. A **17**, 1785 (2000).
[28] D. Mugnai, A. Ranfagni, and R. Ruggeri, Phys. Rev. Lett. **84**, 4830 (2000).
[29] C.J.R. Sheppard, J. Opt. Soc. Am. A **18**, 2594 (2001).
[30] M. Zamboni-Rached, K.Z. Nobrega, H.E. Hernandez-Figueroa, and E. Recami, e-print physics/0209101.
[31] C. Conti, S. Trillo, P. Di Trapani, O. Jedrkiewicz, G. Valiulis, A. Piskarskas, O. Jedrkiewicz, and J. Trull, Phys. Rev. Lett. **90**, 170406 (2003).
[32] S. Trillo, C. Conti, P.D. Trapani, O. Jedrkiewicz, J. Trull, and G. Valiulis, in *Nonlinear Guided Waves and Their Application* (Optical Society of America, Washington, D.C. 2002) paper NLMB.
[33] P.D. Trapani *et al.*, Phys. Rev. Lett. (to be published), e-print physics/0303083.
[34] C. Conti and S. Trillo, Opt. Lett. (to be published), e-print physics/0208097.
[35] L.W. Liou, X.D. Cao, C.J. McKinstrie, and G.P. Agrawal, Phys. Rev. A **46**, 4202 (1992).
[36] E.J. Fonseca, S.B. Cavalcanti, and J.M. Hickmann, Opt. Commun. **169**, 199 (1999).
[37] A. Picozzi and M. Haelterman, Phys. Rev. Lett. **88**, 083901 (2002).
[38] S. Trillo, C. Conti, P.D. Trapani, O. Jedrkiewicz, J. Trull, G. Valiulis, and G. Bellanca, Opt. Lett. **27**, 1451 (2002).
[39] *Spatial Solitons*, edited by S. Trillo and Torruellas (Springer-Verlag, Berlin, 2001) Chap. 6.
[40] A.V. Buriak, P.D. Trapani, D.V. Skryabin, and S. Trillo, Phys. Rep. **370**, 63 (2002).
[41] A. Hasegawa, Opt. Lett. **9**, 288 (1984).
[42] M.A. Porras, S. Trillo, C. Conti, and P.D. Trapani, Opt. Lett. (to be published).
[43] P. Saari, e-print physics/0103054.
[44] V.V. Konotop and M. Salerno, Phys. Rev. A **65**, 021602 (2002).
[45] M. Kramer, L. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **88**, 180404 (2002).
[46] J. Salo and M.M. Salomaa, J. Phys. A **34**, 9319 (2001).
[47] For the sake of concreteness, I use the term X wave, without making distinction among the many families of localized-wave patterns solutions to the linear Helmotz equation. See, e.g., Ref. [43].
[48] It is remarkable that both X waves and MI plane waves, $\exp(i\omega t + ik_x x + ik_y y)$, fill all the space-time and have infinite energy. Furthermore, X waves may be enriched by orthogonality properties as described in Ref. [46].